

Emergence of Deterministic Chaos in Mechanical Systems under Small Periodic External Nonautonomous and Parametric Self-Organizing Bifurcation Perturbations

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Abstract. Beginning with the classical Van der Pol equation, the methods of dynamical systems theory and synergetics have become widely used in the analysis of nonlinear phenomena. The primary objective is to identify stabilization mechanisms that govern the emergence of ordered regimes and self-sustained oscillations, which may either evolve into deterministic chaos or remain stable. These concepts are illustrated using models of forced and self-organizing combustion processes, systems of celestial mechanics, and nonlinear dynamical systems exhibiting finite-time blow-up solutions.

1. Emergence of Deterministic Chaos in Classical Mechanical Systems under Small Nonautonomous Periodic Perturbations

Consider the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) + \mu F_{\text{ext}}, \quad 0 \leq \mu \leq 1, \quad (1)$$

which, when reduced to wave variables $\xi = x - ct$, leads to the situation when $u(x, t) = u(\xi)$, $v(x, t) = v(\xi)$. To which small periodic perturbations F_{ext} are then applied with a period T .

The Duffing Model [1].

The equation

$$x'' + kx' + \omega_0^2 x + ax^3 = F(\omega t), \quad (2)$$

where $k, \omega_0, a, F, \omega$ are constants, and $F \cos(\omega t)$ (i.e. $F_{\text{ext.}}(\omega t)$) denotes the external periodic forcing. The Duffing equation was first investigated by Duffing in 1918 and has since become one of the fundamental models in nonlinear dynamics. It continues to attract considerable attention because deterministic chaotic behavior

of its trajectories has been observed experimentally under appropriate conditions. Equation (2) is equivalent to the following system:

$$\frac{dv}{dt} = -[kv + \omega_0^2 x + ax^3] - F \cos(\omega t), \quad \frac{dx}{dt} = y, \quad (3)$$

where we assume $F \cos(\omega t) = \mu F_{\text{ext.}}(\omega t)$, $0 \leq \mu = 1$. For $\mu = 0$ system (3) on the phase plane $\{x, y\}$ has equilibrium states: $(x = 0, y = 0)$ – saddle, $(x = -\omega_0^2/a, y = 0)$ – center or focus. As well as a loop of separatrices 1 surrounding $(x = -\omega_0^2/a, y = 0)$ and passing through the saddle $(x = 0, y = 0)$. It gives a rough homoclinic structure $\{\Gamma_\mu\}$ in the region $G_{\varepsilon,1} : g_{\varepsilon,1} \otimes \{0 \leq t < \infty\}$, which provides chaotic behavior of trajectories.

The model of Ya. B. Zeldovich and D. A. Frank-Kamenetsky [2] describing the combustion process was studied by means of the wave variable $\xi = x - ct$, which leads to the equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - \delta u(u - \alpha)(u - 1), \quad (4)$$

where $u = u(t, x)$, $\lim_{x \rightarrow -\infty} u = 1$, $\lim_{x \rightarrow +\infty} u = 0$; $\alpha \in (0, 1)$, $\delta > 0$ are parameters, and in the absence of $F_{\text{ext.}}(u, x, t)$ in which the modes of self-oscillations are investigated. If we introduce $F_{\text{ext.}}(u, x, t)$ by the term $\mu \cdot F_{\text{ext.}}(u, x, t)$, $0 \leq \mu = 1$, then for $\mu = 0$, the system on the phase plane $\{u, v = u'_x\}$ has a separatrix contour 1 eye-type structure formed by separatrices $1_1, 1_2$ near the lying saddles $(u = 0, v = 0)$, $(u = 1, v = 0)$, surrounding the center $(u = \alpha, v = 0)$ at $\alpha\delta > 0$, or of the figure-eight type formed by the separatrices $1_1, 1_2$ of the saddle $(u = \alpha, v = 0)$, surrounding the centers on the phase plane $\{u, v\}$ at $\mu = 0$ $(u = 0, v = 0)$, $(u = 1, v = 0)$ when $\alpha\delta < 0$. For system (4) in the domain $G_{\varepsilon,1} : g_{\varepsilon,1} \otimes \{0 \leq t < \infty\}$, we have at $0 \neq |D| = 1$ a rough homoclinic structure $\{\Gamma_\mu\}$, which provides chaotic behavior of trajectories.

Theorem 1. *In the Duffing and Zel'dovich-Frank-Kamenetskii models, in the domain $G_{\varepsilon,1} : g_{\varepsilon,1} \otimes \{0 \leq t < \infty\}$ in the presence of a periodic external force $F_{\text{ext.}}$ ($0 \neq F_{\text{ext.}} = 1$) and a separatrix contour 1, there exists a rough homoclinic structure $\{\Gamma_\mu\}$ corresponding to the chaotic behavior of trajectories.*

2. Deterministic chaos in some models of mechanical systems with small periodic disturbances

Consider the *electric line model* [3] in the form:

$$\frac{\partial U}{\partial t} + v_0 \frac{\partial U}{\partial x} - \gamma \frac{\partial^2 U}{\partial x^2} = -\gamma (U - \beta U^3) + A_\omega \sin(x - ct), \quad c > 0, \quad (5)$$

reduced in the basic form to a system of ordinary differential equations:

$$\frac{du}{d\xi} = v, \quad \frac{dv}{d\xi} = \frac{1}{v} [-\gamma u + (v_0 - c)v + \gamma u^3] + A_\omega \sin \xi. \quad (6)$$

where $\xi = x - ct$, $|A_\omega| = 1$, $A_\omega = A_\omega|_{c=0} = 0$.

In the case $A = A_\omega$, system (6) under condition $v_0 = c$ has two saddles ($u_{1,2} = \pm 1, v = 0$) in the phase plane $\{u, v\}$ and a loop of l -type separatrices surrounding the center $(0, 0)$. Then for $v_0 \neq c$, $0 \neq |A_\omega| = 1$ in the region $G_{\xi,1} : g_{\xi,1} \otimes \{0 \leq \xi < \infty\}$ we have the stochastic mode for the wave model (5).

Theorem 2. *In the power line model (5), if the condition is met $0 \neq |v_0 - c| < \varepsilon$, $0 \neq |A_\omega| = \varepsilon$, there is a rough homoclinic structure $\{G_\omega\}$ in $G_\omega : g_\omega \otimes \{0 \leq \xi < \infty\}$, corresponding to the chaotic behavior of trajectories.*

In [4], the motion of an electron in a one-dimensional system with a displacement phase transition was considered, and we show by qualitative analysis that under certain conditions it is characterized by stochastic wave solutions.

In this case, the equations of motion of the electron are a system of two coupled nonlinear oscillators, the displacement field of which is associated with the electron and has the form

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m_e} \psi_{xx} - g_1 \psi u_x = 0, \quad mu_t - mc_0^2 u_{xx} - |A|u + Bu^3 = g_2 e(|\psi|^2) x, \quad (7)$$

where $\psi(x, t)$ is the wave function of the electron; $u(t, x)$ is the displacement field of the atom; m_e is the mass of the electron; m is the mass of the atom; $0 < g = 1$ [7, § 4.4], where $g_1 = g_2 = g$.

By introducing [5] new variables

$$\xi = x - vt, \quad u = u(\xi), \quad \psi = \varphi(\xi) \left\{ \frac{1}{\hbar} Et + \frac{1}{\hbar} \left(m_e v_1 - \frac{1}{2} m_e v_2 t \right) \right\},$$

corresponding to solutions of the “traveling wave” type, i.e. solitons, we obtain a system second-order ordinary differential equations

$$\frac{\hbar^2}{2m} \frac{d^2 \varphi}{d\xi^2} = -E\varphi + g_1 \varphi \frac{du}{d\xi}, \quad (v^2 - c_0^2) \frac{d^2 u}{d\xi^2} - \frac{|A|}{m} u + \frac{B}{m} u^3 = g_2 \frac{e}{m} \frac{d(\varphi^2)}{d\xi},$$

which corresponds to the system

$$\frac{d\varphi}{d\xi} = \theta, \quad \frac{d\theta}{d\xi} = -M\varphi - g_1 \varphi v, \quad (8)$$

$$\frac{du}{d\xi} = v, \quad \frac{dv}{d\xi} = \frac{1}{m(v^2 - c_0^2)} \left\{ |A|u - Bu^3 + g_2 e \frac{d(\varphi^2)}{d\xi} \right\}, \quad (9)$$

with $\nu - c_0 < 0$, $M = 2mE/\hbar^2$, $\nu \neq c_0$, and a four-dimensional phase space $\{\varphi, \theta, u, w\}$.

Let us perform a qualitative analysis of system (9), assuming first $g_1 = g_2 = 0$, then $g_1 = 0$, $0 < g_2 \leq 1$, and finally $0 < g_{1,2} \leq 1$.

In the case $g_1 = g_2 = 0$, system (8) on the phase plane $\{\varphi, \theta\}$ has a center-type equilibrium state that defines solutions everywhere except $\varphi = \theta = 0$ periodic in ξ ones, which are parametrically expressed as

$$\varphi = r \cos \xi, \quad \theta = \frac{r}{\sqrt{M}} \sin \xi, \quad r > 0. \quad (10)$$

In the same case, the system (9) on the phase plane $\{u, w\}$ has: for $\nu - c_0 < 0$ a closed separatrix contour of the "eye" type, composed of rough saddles ($u = \pm 1, w = 0$) and their separatrices, enclosing the center ($u = 0, w = 0$) (this case for $m = mc_0^2 = |A| = 1$ is considered in [5, section 1.5], where parts $L \cap \{w > 0\}$, $L \cap \{w < 0\}$ of the separatrix L correspond to two types of solitons).

When $\nu - c_0 > 0$ is a closed separatrix contour of the "figure eight" type, composed of loops L_1, L_2 of separatrix L of the coarse saddle ($u = 0, w = 0$), surrounding the centers ($u = 1, w = 0$), ($u = -1, w = 0$) (each L_1 and L_2 has its own soliton solution).

In the case $g_1 = 0, 0 < g_2 \leq 1$, taking into account (10), we have $H = \frac{d(\phi^2)}{d\xi} = H \sin 2\xi$, where $H = \frac{e^2 h^2}{2mE}$, and the system (9) takes the form

$$\frac{du}{d\xi} = w, \quad \frac{dw}{d\xi} = \frac{1}{m(\nu^2 - c_0^2)} \{|A|u - Bu^3 + g_2 H \sin 2\xi\} \quad (11)$$

The system (11) is autonomous with small non-autonomous periodic perturbations in ξ . Applying the research methodology described in [3, 6, 7], the presence for $g_2 = 0$ of separatrix contours L or $L_{1,2}$ in (11) entails for $0 < g_2 \leq 1$ the presence in the extended phase space $\{u, w, \xi\}$ of a rough homoclinic structure. Such a structure is formed by the infinite intersection of two-dimensional separatrix manifolds induced by rough saddles of system (11) at $g_2 = 0$.

Theorem 3. *When $g_1 = 0, 0 < g_2 \leq 1$ in system (9), which in this case is equivalent to system (11), in a three-dimensional phase space $\{\phi, u, w\} \subset \{\phi, \theta, u, w\}$, where ϕ, θ are periodic in ξ , there is a rough homoclinic structure $\{\Gamma_{g_1}\}$ that defines chaotic wave solutions for system (7) at $g_1 = 0, 0 < g_2 \leq 1$.*

3. Chaotic modes and explosive blow-up solutions

As a result of changing the values of the system parameters, which is a manifestation of self-organization, in fact, we come to deterministic chaos. For such a process, the methods of "grafting" and synthesis are applicable [8, 9]. "Grafting" methods assume the presence of a family of periodic trajectories in the region of the system, which transfer (graft) periodicity to another system, which has regions with homoclinic structures with closed separatrix contours.

We proceed to the study of systems with explosive blow-up solutions [9], which are conditioned by sufficient conditions for the presence of a special type P_∞ of Poincare circle, introduced in [10]. The generalized condition of a special type is given in [9], and, in particular, it is related to the presence of two-dimensional subsystems in the initial systems, which are reduced in a simplification of the condition from [9] to the solution of the equation

$$xP_m(x, y) - yQ_m(x, y) = 0, \quad (12)$$

where $P_m(x, y)$ and $Q_m(x, y)$ are the senior members in the initial system

$$\frac{dx}{dt} = \sum_{i=1}^m P_i(x, y), \quad \frac{dy}{dt} = \sum_{i=1}^m Q_i(x, y).$$

Let us further consider the departure of the trajectories of the system under study in P_∞ along directions in a finite time. In this case, there are cases: departure to the Poincare equator, or all of P_∞ , or on its arcs, or on special points P_∞ . Thus, the problem is reduced to finding the roots of the nonlinear equation (12) (x_j, y_j , $j = 1, n + 1$). These roots determine the directions along which the trajectories of the system go in P_∞ in a finite time. Then the Poincare equator P_∞ is not a solution of the dynamical system under study. In the simplest cases, when the trajectories are directed to a point $y = 0$ or to $x = 0$, we find the roots of the equations $xP_m(x, y) = 0$, or $yQ_m(x, y) = 0$, respectively. In an even simpler case, finding roots can be limited to finding the roots of equations $P_m(x, y) = 0$, or $Q_m(x, y) = 0$, respectively. If in all these cases it is possible to identify divergence in directions, then we can assume that deterministic chaos is accompanied by explosive growth.

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